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# Spinor chain path integral for the Dirac equation

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**Abstract.** A path integral that reduces to Feynman's checkerboard rule in one space dimension is found for the retarded Dirac propagator in three space dimensions. The only variables are two-component spinors and a binary chirality variable. No action functional is employed. Each spinor together with a chirality corresponds to a spacetime displacement during a time  $\varepsilon$ . A sequence of spinors and chiralities determines a polygonal spacetime path, and the first and last spinors specify the initial and final spin states. The transition amplitude for a sequence is given by  $\langle \nu_N | \nu_{N-1} \rangle \langle \nu_{N-1} | \dots | \nu_2 \rangle \langle \nu_2 | \nu_1 \rangle (i\varepsilon m)^R$  where  $\nu_j$  are the spinors,  $\langle | \rangle$  is the ordinary inner product in spin space,  $m$  is the electron mass and  $R$  is the number of times the chirality switches. Integrating over all sequences corresponding to a given displacement yields the Dirac propagator in the limit  $\varepsilon \rightarrow 0$ . With  $\varepsilon > 0$  this formulation provides an alternative to the point particle model of the electron.

In an external electromagnetic potential  $A_\mu$  the amplitude for a path  $C$  is multiplied by  $\exp(-ie \int_C A_\mu dx^\mu)$ , requiring spacetime coordinates to specify  $A_\mu(x)$ . The usual perturbation expansion is derived from this rule. These results are extended to non-Abelian gauge potentials. Quantised interaction of particles is not treated.

## 1. Introduction and summary

Since Feynman (1948) introduced the path integral formulation of quantum mechanics various techniques for including the spin degree of freedom have been developed. Some apply to particles, others to fields, and the spin has been treated with variables that are commuting, anticommuting, discrete and continuous. But there is one feature almost all these techniques have in common: the spin is treated as a separate degree of freedom, in addition to spacetime location (see e.g. Schulman 1981 and references therein).

Due to the nature of the Dirac equation, spin and translation are in fact tied together. As a consequence, one can write down path integral formulations of the Dirac propagator which involve *only* spin variables, and it is within this category that the formulation of this paper falls.

The first published work to recognise and exploit the tie between spin and translation appears to be that of Riazanov (1958). By introducing into the Dirac equation a fifth parameter and working with a cleverly chosen 'mixed representation' of spin space he was able to write down a path integral in which the paths are sequences of spin states which alternate between the two representations. Corresponding spacetime paths on a lattice are determined uniquely by a specific rule. The amplitude for a sequence is of the form  $\pm \exp iS$ , where  $S$  bears a resemblance to the classical action functional for a scalar relativistic particle, and the  $\pm$  sign is determined by another rule. These rules have no apparent geometric rationale (in 3+1 dimensions), and the dependence

on the fifth parameter must be integrated out at the end. Nevertheless, Riazanov gives interesting and perspicuous derivations of the non-relativistic and classical limits via his formalism.

Feynman's checkerboard path integral (Feynman and Hibbs 1965) for the Dirac particle in  $1+1$  dimensions is another formulation that exploits the connection between 'spin' and translations. (Actually in one space dimension there is no spin but only chirality.) In the checkerboard picture, there is no fifth parameter, and the chiral (Weyl) representation is used (implicitly) throughout. The paths are sequences of chiralities, right, or left, and correspond directly to paths on a spacetime lattice that step to the right or left at the speed of light in the same sequence. The amplitude for a path is simply  $(i\epsilon m)^R$  where  $\epsilon$  is the step length,  $m$  is the mass, and  $R$  is the number of bends on the path, i.e. reversals of chirality.

The appeal of the checkerboard formulation is that it introduces no extraneous variables, the correspondence between chirality states and spacetime translations is geometrically evident and the amplitude rule is as simple as one could hope for. This simplicity and economy is attained by a *distancing* from the classical formulation, and the connection between the path integral formalism and classical mechanics is lost.

The path integral discussed in this paper constitutes a generalisation of Feynman's checkerboard model to  $3+1$  spacetime dimensions. It is a path integral for the retarded (or advanced) propagator of a Dirac particle that employs as variables *only* two-component spinors and a chirality variable indicating right or left-handedness of the spinors. The germ of this formulation is contained within the fact that the velocity operator  $\alpha$  for the Dirac particle is proportional to the spin operator when it acts separately on right- or left-handed components of the Dirac spinor. This is seen clearly in the chiral (Weyl) representation where  $\alpha$  takes the form

$$\begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$

with  $\sigma$  the Pauli matrices. In this sense velocity is equivalent to spin, with an opposite association for opposite chiralities.

Two independent derivations of the path integral are given in §§ 2.1 and 2.2. Equivalence to the Dirac propagator is demonstrated, and properties of the spinor chain form are examined. In § 2.7 the results are extended to a Dirac particle in an arbitrary number of spatial dimensions.

In § 3 it is explained from a stochastic point of view why in two or more dimensions the step speed must be faster than  $c$ , and the stochastic approach to the light cone is analysed by analogy to an ordinary random walk with drift. The effect of mass is described, and the passage to the non-relativistic limit is explained by analogy with a transition from a non-Markov to a Markov process description.

In § 4 it is shown that the effect of an external gauge potential is to multiply the amplitude for each path  $C$  by the parallel transport operator  $\mathcal{P} \exp(-ig \int_C A_\mu dx^\mu)$  for that path. It is surprising that for a particle with spin the contribution of the potential still factors out in this manner, so in § 4.2 we show how to recover the usual perturbation expansion in which  $A_\mu(x)$  appears only in the combination  $\mathcal{A}(x) = \gamma^\mu A_\mu(x)$ . This combination is seen to represent the effect of the gauge potential integrated over the cone of (infinitesimal) steps leading away from  $x$  toward the future. Finally, in § 5 we speculate on the possibility of extending the spinor chain formulation to quantised interactions, and an alternative to the point particle model of the electron is suggested.

## 2. The spinor chain path integral

This section begins with a derivation that leads directly from the Dirac equation in the chiral (Weyl) representation to the spinor chain path integral. Next, both for computational convenience and because it provides insight into the formalism, we return to the beginning and derive a more general form of the path integral. The path integral is evaluated in terms of the latter form and shown to yield the (retarded) Dirac propagator. The spinor chain form is then motivated and derived from the general form.

### 2.1. Direct route to the spinor chain path integral

The free particle Dirac equation  $\gamma^\mu \partial_\mu \psi = -im \psi$  in the chiral representation

$$\gamma^\mu = \begin{pmatrix} 0 & -\sigma_\pm^\mu \\ -\sigma_\pm^\mu & 0 \end{pmatrix}$$

may be written

$$\sigma_\pm^\mu \partial_\mu \psi_\pm = im \psi_\pm \tag{1}$$

in units where  $\hbar = c = 1$ .  $\psi_\pm$  are two-component spinors corresponding to the upper and lower pairs of components of the Dirac spinor and  $\sigma_\pm^\mu = (\pm \boldsymbol{\sigma}, 1)$  are  $2 \times 2$  matrices, with  $\boldsymbol{\sigma}$  the Pauli matrices. The velocity operator for  $\psi_\pm$  is  $\pm \boldsymbol{\sigma}$ , so that we might say  $\sigma_\pm^\mu$  is the ‘four-velocity operator’. The key to the derivation is a decomposition of  $\sigma_\pm^\mu$  into an overcomplete set of spin projections and it goes as follows.

To each three-vector  $\hat{n}$  on the unit sphere is associated a spin projection operator  $P(\hat{n}) = \frac{1}{2}(1 + \hat{n} \cdot \boldsymbol{\sigma})$ . Integrating over the sphere we have

$$\frac{1}{3} \boldsymbol{\sigma} = \int (d\Omega/2\pi) P(\hat{n}) \hat{n}$$

which is verified using  $\int d\Omega \hat{n} = 0$  and  $\int (d\Omega/4\pi) \hat{n}^i \hat{n}^j = \frac{1}{3} \delta^{ij}$ . Now  $P(\hat{n})$  may be written as  $|\nu\rangle\langle\nu|$  where  $|\nu\rangle$  is a unit spinor satisfying  $P(\hat{n})|\nu\rangle = |\nu\rangle$  or, equivalently,  $\hat{n} = \langle\nu|\boldsymbol{\sigma}|\nu\rangle$ . Since  $\int (d\Omega/2\pi) P(\hat{n}) = 1$  we have the identity

$$\sigma_\pm^\mu = \int (d\Omega/2\pi) |\nu\rangle\langle\nu| n_\pm^\mu \tag{2}$$

with  $n_\pm = (\pm 3\langle\nu|\boldsymbol{\sigma}|\nu\rangle, 1)$ . Note that  $n_\pm$  is *not* a null vector, so this is not the usual association of two-component spinors with null spacetime vectors. Rather,  $n_\pm$  is a displacement during a unit time at speed  $3c$ , relative to the arbitrary fixed inertial coordinate system we are employing.

Consider first the massless case.  $\psi_\pm$  then satisfies the Weyl equation  $\sigma_\pm^\mu \partial_\mu \psi_\pm = 0$ . Substituting for  $\sigma_\pm^\mu$  from (2) and multiplying by  $\varepsilon$  we obtain

$$\int (d\Omega/2\pi) |\nu\rangle\langle\nu| (\varepsilon n_\pm^\mu \partial_\mu \psi_\pm) = 0.$$

Approximating the derivative by a finite difference

$$\varepsilon n_\pm^\mu \partial_\mu \psi_\pm = \psi_\pm(x) - \psi_\pm(x - \varepsilon n_\pm) + O(\varepsilon^2),$$

the Weyl equation becomes

$$\psi_{\pm}(x) = \int (d\Omega/2\pi) |\nu\rangle\langle\nu| \psi_{\pm}(x - \epsilon n_{\pm}) + O(\epsilon^2).$$

Thus according to the Weyl equation, as  $\epsilon \rightarrow 0$  the matrix  $|\nu\rangle\langle\nu|$  serves to propagate the spinor amplitude  $\psi_{\pm}$  from  $x - n_{\pm}$  to  $x$ . Hence we conjecture that the retarded propagator  $K^{\pm}$  can be written as  $\lim_{\epsilon \rightarrow 0} K_{\epsilon}^{\pm}$  with

$$K_{\epsilon}^{\pm}(x, t; x', t') = \int \prod_{i=1}^N (d\Omega_i/2\pi) |\nu_N\rangle\langle\nu_N| \dots |\nu_1\rangle\langle\nu_1|, \tag{3}$$

where the integration is over those chains of  $N = (t - t')/\epsilon$  projections satisfying the constraints  $\pm 3\epsilon \sum_{i=1}^N \langle\nu_i|\sigma|\nu_i\rangle = x - x'$  or, equivalently,

$$\sum_{i=1}^N |\nu_i\rangle\langle\nu_i| = (1/2\epsilon)[(t - t') \pm \frac{1}{3}(x - x') \cdot \sigma]. \tag{4}$$

Note that the approximation  $\epsilon n^{\mu} \partial_{\mu} \psi = \psi(x + \epsilon n) - \psi(x) + O(\epsilon^2)$  would have led instead to the *advanced* propagator.

In the massive case the Dirac equation may be written as

$$\psi_{\pm}(x) = \int (d\Omega/2\pi) |\nu\rangle\langle\nu| [\psi_{\pm}(x - \epsilon n_{\pm}) + i\epsilon m \psi_{\pm}(x - \epsilon n_{\pm})] + O(\epsilon^2),$$

since  $i\epsilon m \psi_{\pm}(x) = i\epsilon m \psi_{\pm}(x - \epsilon n_{\pm}) + O(\epsilon^2)$ . Thus, as in the massless case, as  $\epsilon \rightarrow 0$  the matrix  $|\nu\rangle\langle\nu|$  propagates the spinor amplitude  $\psi$  from  $x - \epsilon n_{\pm}$  to  $x$ . In addition, a switch in chirality is accompanied by a factor  $i\epsilon m$ . Hence we conjecture that, by attaching a chirality variable  $\chi_i = \pm 1$  to each link  $|\nu_i\rangle\langle\nu_i|$  in a chain, the retarded propagator  $K_{\chi_N, \chi_0}$  connecting a state of chirality  $\chi_0$  to one of chirality  $\chi_N$  can be written as  $\lim_{\epsilon \rightarrow 0} [K_{\epsilon}]_{\chi_N, \chi_0}$  with

$$[K_{\epsilon}(x, t; x', t')]_{\chi_N, \chi_0} = \int \prod_{i=1}^N (d\Omega_i/2\pi) \sum_{\chi_1, \dots, \chi_{N-1}} |\nu_N\rangle\langle\nu_N| \dots |\nu_1\rangle\langle\nu_1| (i\epsilon m)^R, \tag{5}$$

where the integration/sum is over all sequences of  $N$  chiralities and projections satisfying the constraint

$$3\epsilon \sum_{i=1}^N \chi_i \langle\nu_i|\sigma|\nu_i\rangle = x - x', \tag{6}$$

and  $R$  is the number of chirality switches along the chain (including a possible switch from  $\chi_0$  to  $\chi_1$ ).

The results of §§ 2.2–2.4 demonstrate that this conjectured representation for the propagator is indeed correct.

### 2.2. The general form of the path integral

We now derive a form of the path integral that works with four-component (Dirac) spinors, employing an arbitrary representation for the Dirac matrices and an arbitrary step speed.

The Dirac equation for a free particle may be written

$$\gamma^{\mu} \partial_{\mu} \psi = -im \psi \tag{7}$$

in units where  $\hbar = c = 1$ .  $\psi$  is a four-component spinor and the  $\gamma^\mu$  are  $4 \times 4$  matrices satisfying  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} = 2 \text{diag}(1, -1, -1, -1)$ . In any particular coordinate system only four derivatives are involved in (7), so that approximating the derivatives  $\partial_\mu \psi$  by finite differences would yield an equation relating  $\psi(x)$  to the values of  $\psi$  at four nearby points.

Our strategy is to rewrite (7) so that  $\psi(x)$  is related instead to all the values of  $\psi$  on a small sphere of nearby points at a slightly earlier time. The method is simplest to see in three-dimensional vector calculus. Instead of writing the gradient of a function  $\phi$  as  $\partial_i \phi$ , which involves just three derivatives of  $\phi$ , it may be expressed as a vector average of the directional derivatives in all directions by

$$\partial^i \phi(x) = \left( (3/4\pi) \int d\Omega \hat{n}^i \hat{n}^j \right) \partial_j \phi(x)$$

where  $\hat{n}$  is a unit vector and the integration is over all vectors on the unit sphere. In other words,  $\partial_i \phi$  has merely been multiplied by the resolution of unity

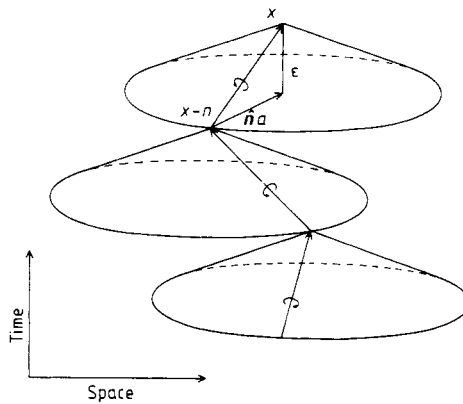
$$\delta^{ij} = (3/4\pi) \int d\Omega \hat{n}^i \hat{n}^j.$$

In the spacetime case we seek a similar resolution of  $\delta_\mu^\nu$ , built with the sphere of four-vectors of the form  $n^\mu = (\hat{n}_a, \epsilon)$  (see figure 1). It is easily verified that

$$\delta_\mu^\nu = (-3/a^2) \Sigma n_\mu n^\nu + (3/a^2 + 1/\epsilon^2) \Sigma' n'_\mu \Sigma n^\nu \tag{8}$$

where  $\Sigma$  stands for the normalised integral  $\int d\Omega/4\pi$  over the sphere of unit vectors  $\hat{n}$ , and the primes indicate an independent sum over the same sphere. (In verifying (8) note that  $\Sigma n^\nu = \epsilon \delta_0^\nu$  and  $\Sigma n_0 n^\nu = \epsilon^2 \delta_0^\nu$ .) Inserting (8) in the Dirac equation (7) we find

$$\begin{aligned} -im\psi &= \gamma^\mu \delta_\mu^\nu \partial_\nu \psi \\ &= \gamma^\mu \left( (-3/a^2) \Sigma n_\mu n^\nu + (3/a^2 + 1/\epsilon^2) \Sigma' n'_\mu \Sigma n^\nu \right) \partial_\nu \psi \\ &= \Sigma [(1/\epsilon)\gamma^0 - (3/a)\gamma^i \hat{n}_i] n^\nu \partial_\nu \psi \end{aligned}$$



**Figure 1.** Spacetime diagram of the last three steps in a typical path leading to  $x$ . The rim of each cone represents a sphere of radius  $a$  centred at the spatial position of the vertex. The vertex occurs a time  $\epsilon$  to the future of the rim, so the speed of a step along the cone is  $a/\epsilon$ . The arrow circling each step represents an associated spin whose chirality is indicated by the sense of circulation (cf §§ 2.1, 2.4).

( $i = 1, 2, 3$ ), which may be rewritten, multiplying both sides by  $-\varepsilon\gamma^0$  and using the notations  $\gamma^0 = \beta$ ,  $\gamma^0\gamma^i = \alpha^i$ ,  $\varepsilon/a = \eta$ , as

$$i\varepsilon m\beta\psi = \sum (1 + 3\eta\alpha \cdot \hat{n})(-n^\nu \partial_\nu \psi). \tag{9}$$

We now approximate the derivative in (9) by a finite difference

$$-n^\nu \partial_\nu \psi(x) = \psi(x - n) - \psi(x) + O(\varepsilon^2) \tag{10}$$

where it is assumed that  $O(\varepsilon) = O(a)$ .  $x - n$  stands for the four-vector  $(x - \hat{n}a, x^0 - \varepsilon)$ .

Substituting (10) in (9) we obtain

$$(1 + i\varepsilon m\beta)\psi(x) = \sum (1 + 3\eta\alpha \cdot \hat{n})\psi(x - n) + O(\varepsilon^2)$$

or, since  $(1 - i\varepsilon m\beta)(1 + i\varepsilon m\beta) = 1 + \varepsilon^2 m^2 = 1 + O(\varepsilon^2)$ ,

$$\psi(x) = (1 - i\varepsilon m\beta) \sum (1 + 3\eta\alpha \cdot \hat{n})\psi(x - n) + O(\varepsilon^2). \tag{11}$$

Equation (11) indicates that, according to the Dirac equation, as  $\varepsilon \rightarrow 0$  the matrix

$$M_\varepsilon(\hat{n}) = (1 - i\varepsilon m\beta)(1 + 3\eta\alpha \cdot \hat{n}) \tag{12}$$

propagates the spinor amplitude from  $x - n$  to  $x$ . We call  $M_\varepsilon(n)$  the *propagation matrix*. Now for each  $n$ ,  $\psi(x - n)$  may in turn be expressed as an integral similar to (11) over a sphere centred at the spatial part of  $x - n$  at time  $2\varepsilon$  to the past of  $x$ , and so on, so that going  $N$  steps into the past we obtain an expression for  $\psi(x)$  which is a multiple integral over all sequences of steps leading to  $x$  (cf figure 1) with an integrand of the form

$$M_\varepsilon(\hat{n}_N) \dots M_\varepsilon(\hat{n}_1)\psi(x_1).$$

The retarded propagator  $K(x, t; x', t')$  can be thought of as the function that results from evolving an initial  $\delta$  function, which suggests that it might be obtained as  $\lim_{\varepsilon \rightarrow 0} K_\varepsilon$  with

$$K_\varepsilon(x, t; x', t') = \int \prod_{i=1}^N (d\Omega_i/4\pi) M_\varepsilon(\hat{n}_N) \dots M_\varepsilon(\hat{n}_1), \tag{13}$$

where  $N = (t - t')/\varepsilon$  and the integration is constrained to those sequences of  $N$  unit vectors satisfying

$$\sum_{i=1}^N \hat{n}_i a = x - x'. \tag{14}$$

Whether this representation of  $K$  is correct or not depends on how the errors, made by neglecting terms of  $O(\varepsilon^2)$  at each step, accumulate in the limit  $\varepsilon, a \rightarrow 0, N \rightarrow \infty$ , with  $N\varepsilon$  and  $a/\varepsilon$  held constant.

In § 2.3 it is seen that this finite difference approximation will converge if and only if the step speed  $a/\varepsilon$  is  $\geq 3^{1/2}c$ . For  $a/\varepsilon > 3^{1/2}c$  we recover the Dirac propagator by evaluating the path integral (13) in the limit  $\varepsilon \rightarrow 0$ .

### 2.3. Evaluation of the path integral and equivalence to the Dirac propagator

The task of performing the integral (13) is similar to that encountered in the problem of random flights (see e.g. Chandrasekhar 1943) despite the difference in physical interpretation. From a stochastic point of view, the most significant difference is that

in our case it is the step speed  $a/\epsilon$  rather than the diffusion constant  $a^2/\epsilon$  that is held fixed as  $\epsilon, a \rightarrow 0$ . Consequently the width of the characteristic Gaussian distribution goes to zero in our limit, and all the physics is to be found in the ‘drift’ that results from a correlation in direction for successive steps. Stochastic aspects of our problem are discussed further in § 3.

The constraint (14) can be incorporated into the measure for the integral (13) by use of a  $\delta$  function, so that the measure becomes

$$\prod_{j=1}^N (d\Omega_j/4\pi) \delta\left(\mathbf{x} - \mathbf{x}' - \sum_{i=1}^N \hat{\mathbf{n}}_{i,a}\right). \tag{15}$$

Writing the  $\delta$  function as a Fourier integral we have

$$\begin{aligned} K_\epsilon(\mathbf{x}, t; \mathbf{x}', t') &= (8\pi^3)^{-1} \int d^3\kappa e^{i\kappa \cdot (\mathbf{x} - \mathbf{x}')} \int \prod_{j=1}^N (d\Omega_j/4\pi) \\ &\quad \times \exp\left(-i \sum_{i=1}^N \kappa \cdot \hat{\mathbf{n}}_{i,a}\right) M_\epsilon(\hat{\mathbf{n}}_N) \dots M_\epsilon(\hat{\mathbf{n}}_1) \\ &= (8\pi^3)^{-1} \int d^3\kappa e^{i\kappa \cdot (\mathbf{x} - \mathbf{x}')} \{A_\epsilon(\boldsymbol{\kappa})\}^N \end{aligned} \tag{16}$$

where

$$A_\epsilon(\boldsymbol{\kappa}) = (4\pi)^{-1} \int d\Omega e^{-i\boldsymbol{\kappa} \cdot \hat{\mathbf{n}}_a} M_\epsilon(\hat{\mathbf{n}}). \tag{17}$$

Substituting for  $M_\epsilon(\hat{\mathbf{n}})$  from (12) in (17) we integrate to obtain

$$A_\epsilon(\boldsymbol{\kappa}) = (1 - i\epsilon m\beta)(j_0(\kappa a) - i3\eta j_1(\kappa a)\boldsymbol{\alpha} \cdot \hat{\boldsymbol{\kappa}}), \tag{18}$$

where  $\boldsymbol{\kappa} = \|\boldsymbol{\kappa}\|$ ,  $\hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}/\kappa$  and  $j_0, j_1$  are spherical Bessel functions of the first kind

$$\begin{aligned} j_0(z) &= \sin z/z = 1 - z^2/3! + z^4/5! - \dots, \\ j_1(z) &= (\sin z - z \cos z)/z^2 = 2z/3! - 4z^3/5! + \dots \end{aligned}$$

$A_\epsilon(\boldsymbol{\kappa})$  is called the *amplification matrix*.

From (16) we see that the  $\epsilon \rightarrow 0$  limit of  $K_\epsilon$  is determined by the  $\epsilon \rightarrow 0$  limit of  $\{A_\epsilon(\boldsymbol{\kappa})\}^N$  (with  $N\epsilon = t - t'$ ). In order for our finite difference scheme to be stable and convergent to the exact propagator it is necessary that the norm of this matrix be bounded for all vectors  $\boldsymbol{\kappa}$  in the limit  $\epsilon \rightarrow 0, N \rightarrow \infty$ . It is here that the restriction on the step speed  $a/\epsilon = \eta^{-1}$  becomes evident.

Consider first the case  $m = 0$ . Then by diagonalising  $\boldsymbol{\alpha} \cdot \hat{\boldsymbol{\kappa}}$ ,  $A_\epsilon(\boldsymbol{\kappa})$  is diagonalised. Since  $(\boldsymbol{\alpha} \cdot \hat{\boldsymbol{\kappa}})^2 = 1$ , the eigenvalues of  $\boldsymbol{\alpha} \cdot \hat{\boldsymbol{\kappa}}$  are  $\pm 1$  so the eigenvalues of  $A_\epsilon(\boldsymbol{\kappa})$  are  $j_0(\kappa a) \pm i3\eta j_1(\kappa a)$ , with squared modulus  $\xi(\kappa a) = j_0^2(\kappa a) + 9\eta^2 j_1^2(\kappa a) = 1 - (\frac{1}{3} - \eta^2)(\kappa a)^2 + \dots$ . Now  $\xi(\kappa a) \leq 1$  for all  $\kappa$  if and only if  $\eta^2 \leq \frac{1}{3}$  (to check this note that  $\xi(0) = 1$ , and for  $\eta^2 = \frac{1}{3}$  the derivative of  $\xi$  is negative out to the first zero of  $j_1$ , which occurs at  $\sim 4.5$ ; beyond 4.5, crude estimates ensure  $\xi < 1$ ). Thus we have stability only if  $\eta^{-1} = a/\epsilon \geq \sqrt{3}$ .

When the mass is non-zero the same condition for stability results, since the effect of the mass term is to add a matrix of order  $\epsilon$  which only changes the norm of  $\{A_\epsilon(\boldsymbol{\kappa})\}^N$  by a bounded factor since  $N\epsilon = t - t'$  is fixed.

The condition  $\eta^2 < \frac{1}{3}$  is sufficient for  $K_\epsilon$  to converge to the exact propagator, as we shall now demonstrate.



Multiplying out the factors and expanding the Bessel functions in (18) we have

$$A_\epsilon(\boldsymbol{\kappa}) = 1 - i\eta a \boldsymbol{\alpha} \cdot \boldsymbol{\kappa} - i\epsilon m \beta - (\kappa a)^2/6 + O(\epsilon \kappa a; (\kappa a)^3) \\ = \exp[-i\epsilon(\boldsymbol{\alpha} \cdot \boldsymbol{\kappa} + \beta m) - (\frac{1}{3} - \eta^2)(\kappa a)^2/2 + O(\epsilon^2; \epsilon \kappa a; (\kappa a)^3)]$$

so that

$$\{A_\epsilon(\boldsymbol{\kappa})\}^N = \exp\{-iN\epsilon(\boldsymbol{\alpha} \cdot \boldsymbol{\kappa} + \beta m) - (\frac{1}{3} - \eta^2)N(\kappa a)^2/2 + N[O(\epsilon^2; \epsilon \kappa a; (\kappa a)^3)]\} \tag{19}$$

If  $\eta^2 < \frac{1}{3}$ , the term in  $N(\kappa a)^2$  acts as a damping factor and only  $\boldsymbol{\kappa}$ 's for which  $\kappa a \leq O(N^{-1/2})$  contribute significantly to the integral in (16). (That the higher-order terms remain small for all values of  $\boldsymbol{\kappa}$  can be seen from the eigenvalues of  $A_\epsilon(\boldsymbol{\kappa})$ .) Together with  $\epsilon \sim O(1/N)$ , this shows that when  $N$  is large we may drop the last term in the exponential without altering the value of the integral in the limit.

Substituting for  $\{A_\epsilon(\boldsymbol{\kappa})\}^N$  from (19) in (16) and dropping the higher-order terms, we obtain for  $K_\epsilon$  for small values of  $\epsilon$  the expression

$$K_\epsilon(\mathbf{x}, t; \mathbf{x}', t') = (8\pi^3)^{-1} \int d^3\kappa e^{i\boldsymbol{\kappa} \cdot (\mathbf{x} - \mathbf{x}')} \exp[-iN\epsilon(\boldsymbol{\alpha} \cdot \boldsymbol{\kappa} + \beta m) - (\frac{1}{3} - \eta^2)N(\kappa a)^2/2]. \tag{20}$$

Noting that the exponential can be written as a product of exponentials, we may invoke the convolution theorem for Fourier transforms to write the integral in (20) as the convolution of a normalised Gaussian of variance  $(\frac{1}{3} - \eta^2)Na^2$  with the expression

$$K(\mathbf{x}, t; \mathbf{x}', t') = (8\pi^3)^{-1} \int d^3\kappa e^{i\boldsymbol{\kappa} \cdot (\mathbf{x} - \mathbf{x}')} \exp[-i(t - t')(\boldsymbol{\alpha} \cdot \boldsymbol{\kappa} + \beta m)],$$

which is independent of  $\epsilon$  (except that  $t - t'$  must be an integer multiple of  $\epsilon$ ) and is in fact the exact retarded propagator for the Dirac equation written as a Fourier transform. As  $\epsilon \rightarrow 0$ , the normalised Gaussian approaches a  $\delta$  function. We have thus demonstrated that for  $\eta^2 < \frac{1}{3}$ ,  $K_\epsilon$  approaches the exact propagator  $K$  as  $\epsilon \rightarrow 0$ .

#### 2.4. Spinor chain form of the path integral

The most significant feature of Feynman's original path integral formulation of quantum mechanics is that it deals directly with probability amplitudes for spacetime processes. In this respect our path integral for the Dirac electron (equations (12), (13), (14)) is somewhat unsatisfactory, since it determines a matrix of amplitudes. To obtain an individual amplitude we must first specify initial and final spin states  $\psi_i, \psi_f$ ; then the matrix element  $\langle \psi_f | K(\mathbf{x}, x') | \psi_i \rangle$  gives the amplitude for a transition from  $\psi_i$  at  $x'$  to  $\psi_f$  at  $x$ .

This situation is sometimes explained by saying that the spin is an 'internal', non-classical degree of freedom which cannot be described in terms of spacetime variables. From this point of view there appears to be no alternative but to consider paths in the direct product of spacetime with an internal spin space if one wants a path integral that deals directly with amplitudes.

There is an alternative, however. Namely, one may describe the spacetime translation degree of freedom in terms of spin. In view of the form of the path integral (13) it will be sufficient to describe the unit sphere of spatial vectors in spinor language, since any spacetime translation can be built from a sequence of unit vectors.

We associate unit vectors with two-component spinors as follows. Each unit vector  $\hat{n}$  determines a unique spin state whose quantisation axis is  $\hat{n}$ . The relation is neatly expressed by writing the spin projection operator  $P(\hat{n}) = \frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \hat{n})$  as  $|\nu\rangle\langle\nu|$  where  $|\nu\rangle$  is a normalised eigenvector of  $P(\hat{n})$  corresponding to the eigenvalue 1. Equivalently, we can write  $\hat{n} = \langle\nu|\boldsymbol{\sigma}|\nu\rangle$ .

In order to implement this association we use the chiral representation for Dirac's equation (7) since spinors of opposite chiralities must separately be associated with spacetime displacements. The chiral representation is given by

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{21}$$

where  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the Pauli matrices. In this representation, the zero-mass propagation matrix (cf (12)) takes the form

$$\begin{pmatrix} 1 + 3\eta\boldsymbol{\sigma} \cdot \hat{n} & 0 \\ 0 & 1 - 3\eta\boldsymbol{\sigma} \cdot \hat{n} \end{pmatrix}.$$

With  $\eta = \frac{1}{3}$  this may be expressed as

$$2 \begin{pmatrix} |\nu\rangle\langle\nu| & 0 \\ 0 & |\tilde{\nu}\rangle\langle\tilde{\nu}| \end{pmatrix} \tag{22}$$

where  $|\nu\rangle$  and  $|\tilde{\nu}\rangle$  are orthogonal unit spinors ( $\langle\nu|\tilde{\nu}\rangle = 0$ ) satisfying  $|\nu\rangle\langle\nu| = \frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \hat{n})$  and  $|\tilde{\nu}\rangle\langle\tilde{\nu}| = \frac{1}{2}(1 - \boldsymbol{\sigma} \cdot \hat{n})$ .

The choice  $\eta = \frac{1}{3}$  satisfies the stability requirement  $\eta^2 < \frac{1}{3}$ , so we know it will yield the correct propagator. Thus according to (22) and (13), the retarded propagator for a right-handed Weyl neutrino can be written as the  $\epsilon \rightarrow 0$  limit of

$$K_\epsilon^+(x, t; x', t') = \int \prod_{i=1}^N (d\Omega_i/2\pi) |\nu_N\rangle\langle\nu_N| \nu_{N-1}\rangle\langle\nu_{N-1}| \dots |\nu_2\rangle\langle\nu_2| \nu_1\rangle\langle\nu_1| \tag{23}$$

where the factor 2 in (22) has been absorbed into the measure. The integration is constrained to those chains of  $N = (t - t')/\epsilon$  spinor projections satisfying  $3\epsilon \sum_{i=1}^N \langle\nu_i|\boldsymbol{\sigma}|\nu_i\rangle = x - x'$ . The propagator for a left-handed neutrino is given by the same formula as (23), but with the reflected relation between spinors and vectors, so that the chains must satisfy  $-3\epsilon \sum_{i=1}^N \langle\nu_i|\boldsymbol{\sigma}|\nu_i\rangle = x - x'$ . We have thus recovered formulae (3) and (4) of § 2.1.

In the massive case the propagation matrix (12) becomes

$$2 \begin{pmatrix} |\nu\rangle\langle\nu| & i\epsilon m |\tilde{\nu}\rangle\langle\tilde{\nu}| \\ i\epsilon m |\nu\rangle\langle\nu| & |\tilde{\nu}\rangle\langle\tilde{\nu}| \end{pmatrix}.$$

The propagator for this case can be expressed purely in terms of two-component spin projections as follows. We attach to each propagation matrix chirality indices  $\chi = \pm 1$ , so that for each pair of values  $\chi, \chi'$ ,  $[M_\epsilon(\hat{n})]_{\chi\chi'}$  is a  $2 \times 2$  matrix. For example,  $M_{+-} = 2i\epsilon m |\tilde{\nu}\rangle\langle\tilde{\nu}|$ . With this notation, the path integral (13) can be written as

$$\begin{aligned} & \int \Pi(d\Omega_i/4\pi) \sum_{\chi_2, \dots, \chi_N} [M_\epsilon(\hat{n}_N)]_{\chi_N \pm 1 \chi_N} \dots [M_\epsilon(\hat{n}_1)]_{\chi_2 \chi_1} \\ & = \int \Pi(d\Omega_i/2\pi) \sum_{\chi_2, \dots, \chi_N} |\nu_N\rangle\langle\nu_N| \dots |\nu_1\rangle\langle\nu_1| (i\epsilon m)^R \end{aligned}$$

where a tilde should appear over  $|\nu_i\rangle\langle\nu_i|$  if  $\chi_i = -1$ , and  $R$  is the number of off-diagonal matrix elements, that is,

$$R = \sum_{i=1}^N \frac{1}{2}(1 - \chi_{i+1}\chi_i). \tag{24}$$

The tildes can be eliminated by a change of variables in the integral. For each term with  $\chi_i = -1$  we integrate over  $-\hat{n}_i$  instead of  $\hat{n}_i$ , leaving off the tildes in the integrand but reversing the direction of the corresponding unit vectors in the constraint (14). With these variables the propagator connecting an initial state of chirality  $\chi_N$  to a final state of chirality  $\chi$  is given by the  $\varepsilon \rightarrow 0$  limit of

$$[K_\varepsilon(\mathbf{x}, t; \mathbf{x}', t')]_{\chi_N \chi_1} = \int \Pi(d\Omega_i/2\pi) \sum_{\chi_2, \dots, \chi_N} |\nu_N\rangle\langle\nu_N| \dots |\nu_1\rangle\langle\nu_1| (i\varepsilon m)^R \tag{25}$$

where the integration/sum is over all sequences of  $N = (t - t')/\varepsilon$  chiralities and spin projections satisfying the constraint

$$3\varepsilon \sum_{i=1}^N \chi_i \langle\nu_i|\boldsymbol{\sigma}|\nu_i\rangle = \mathbf{x} - \mathbf{x}', \tag{26}$$

and  $R$  is the number of times the chirality switches along the chain (including a possible switch from  $\chi_N$  to  $\chi_{N+1}$ . The possibility of such a final switch does not affect the propagator in the limit  $\varepsilon \rightarrow 0$ , however, so if initial–final symmetry is desired in (15) we may fix  $\chi_N$  and use it as the first index on  $K_\varepsilon$ ). We have thus recovered formulae (5) and (6) of § 2.1.

Finally, note that different forms for the mass term in (25) arise from different representations of  $\beta$ . For example, if  $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  we have  $\pm \varepsilon m$ , with  $+$  for left–right switches and  $-$  for right–left.

### 2.5. The spinor measure

A remarkable feature of the spinor chain form (25) for the free particle Dirac propagator is that spacetime coordinates do not enter. The amplitude, the constraint (26) on the spinor chains and the measure are all expressed purely in terms of unit two-component spinors and chiralities. We now elaborate on this point regarding the measure.

The rays in spin space comprise  $\mathbb{C}P^1$ , the projective space of  $\mathbb{C}^2$ .  $\mathbb{C}P^1$  has the topology of a two-sphere and we have made use of a particular correspondence between this and the unit sphere in space. If  $|\nu\rangle\langle\nu|$  is the ray determined by a unit spinor  $\nu = e^{i\phi}(\cos \frac{1}{2}\theta e^{-i\phi/2}, \sin \frac{1}{2}\theta e^{i\phi/2})$ , the corresponding unit vector  $\hat{n} = \langle\nu|\boldsymbol{\sigma}|\nu\rangle$  is given in spherical coordinates by  $(\theta, \phi)$ .

In the derivation of § 2.2, the choice of  $d\Omega/4\pi$  as the measure of one elementary step was tacitly motivated by the invariance of the Dirac equation under spatial rotations and the normalisation  $\int d\Omega/4\pi = 1$ . In spin space these rotations correspond to unitary transformations, and in fact the measure  $d\Omega/2\pi$  can be characterised as the unique unitary-invariant measure on  $\mathbb{C}P^1$  satisfying  $\int_{\mathbb{C}P^1} |\nu\rangle\langle\nu| = 1$ . It is thus not necessary to refer to properties of spacetime to motivate the choice of  $d\Omega/2\pi$ . One need only demand invariance under unitary transformations, a natural requirement since it is precisely these transformations that preserve the amplitude for a spin chain.

Finally we mention that since unit spinors related by a simple phase factor determine the same projection, one may divide the measure by  $2\pi$  and integrate over unit spinors rather than rays.

2.6. Geometrical interpretation of the spinor amplitude

The amplitude for a spinor  $\nu$  to be followed by  $\nu'$  in the path integral (25) is  $\langle \nu' | \nu \rangle$ . We shall now demonstrate that  $\langle \nu' | \nu \rangle = \cos \frac{1}{2}\omega e^{i\zeta}$ , where  $\omega$  is the angle between the corresponding unit vectors and  $\zeta$  is a non-geometrical phase angle. This result will be used in § 3 where we explore the analogy of propagation of a Dirac particle to a random walk with drift.

The squared modulus  $|\langle \nu' | \nu \rangle|^2$  is given by  $\langle \nu' | \nu \rangle \langle \nu | \nu' \rangle = \langle \nu' | \frac{1}{2}(1 + \hat{n} \cdot \sigma) | \nu' \rangle = \frac{1}{2}(1 + \hat{n} \cdot \hat{n}') = \frac{1}{2}(1 + \cos \omega) = \cos^2 \frac{1}{2}\omega$ , where  $\omega$  is the angle between the unit vectors  $\hat{n} = \langle \nu | \sigma | \nu \rangle$  and  $\hat{n}' = \langle \nu' | \sigma | \nu' \rangle$ . Thus  $\langle \nu' | \nu \rangle$  has the form  $\cos \frac{1}{2}\omega e^{i\zeta}$  for some  $\zeta$ . Since  $\zeta$  depends on the overall phase of the spinors it does not appear to have a geometrical interpretation. Moreover, suppose we fix the phase using a standard parametrisation  $\nu = (\cos \frac{1}{2}\theta e^{-i\phi/2}, \sin \frac{1}{2}\theta e^{i\phi/2})$  for the spinors. Then we have

$$\langle \nu' | \nu \rangle = \cos \frac{1}{2}\theta' \cos \frac{1}{2}\theta e^{-i(\phi' - \phi)/2} + \sin \frac{1}{2}\theta' \sin \frac{1}{2}\theta e^{i(\phi' - \phi)/2}.$$

When  $\phi' = \phi$ ,  $\langle \nu' | \nu \rangle = \cos \frac{1}{2}(\theta' - \theta) = \cos \frac{1}{2}\omega$ , so that  $\zeta = 0$ . In general,  $\zeta$  can be written in terms of the spherical angles  $\theta, \phi, \theta', \phi'$ ; it is not invariant under rotations and is a discontinuous function of the unit vectors  $\hat{n}, \hat{n}'$  as can be seen by choosing special values for the angles.

2.7. The Dirac particle in  $d$  space dimensions

Insight into our formalism is obtained by examining also the cases of one and two spatial dimensions, for which we now prepare the way. The derivation of the path integral in § 2.2 can be repeated for the case of an arbitrary space dimensionality  $d$ . There results the propagation matrix (cf (12))

$$M_\varepsilon(\hat{n}) = (1 - i\varepsilon m\beta)(1 + d\eta\alpha \cdot \hat{n}). \tag{27}$$

In one spatial dimension we have for the massless amplification matrix (cf (17))

$$A(\kappa) = \frac{1}{2} \sum_{n=\pm 1} e^{-i\kappa n a} (1 + \eta\alpha n) = \cos \kappa a - i\eta\alpha \sin \kappa a.$$

In two dimensions we have

$$A(\kappa) = (2\pi)^{-1} \int_0^{2\pi} d\phi e^{-i\kappa \cdot \hat{n} a} (1 + 2\eta\alpha \cdot \hat{n}) = J_0(\kappa a) - i2\eta J_1(\kappa a) \alpha \cdot \hat{\kappa},$$

where  $\kappa = \|\kappa\|$  and  $J_0, J_1$  are cylindrical Bessel functions of the first kind, with the expansions

$$J_0(z) = 1 - z^2/4 + \dots, \quad J_1(z) = z/2 - z^3/16 + \dots$$

(For a space of arbitrary dimension  $d \geq 2$  the result is

$$A(\kappa) = \Gamma(\nu + 1)(2/\kappa a)^\nu [J_\nu(\kappa a) - id\eta J_{\nu+1}(\kappa a) \alpha \cdot \hat{\kappa}]$$

where  $\nu = \frac{1}{2}(d-2)$  and  $J_\nu, J_{\nu+1}$  are Bessel functions of the first kind. The term in  $\alpha$  is obtained by differentiating the first term with respect to  $\kappa$  and using a recursion relation for the Bessel function.)

By an analysis similar to that which follows (18) we see that the stability condition is  $\eta^2 \leq 1$  for  $d=1$  and  $\eta^2 \leq \frac{1}{2}$  for  $d=2$ . The evaluation of the path integral is slightly different in the one-dimensional case, since the contribution to the integral corresponding to (16) comes not only from  $\kappa a$  near zero but also from near integer multiples of  $\pi$ . This can be seen from the squared modulus of the eigenvalues,  $\cos^2 \kappa a + \eta^2 \sin^2 \kappa a$ , and is connected to the fact that in one dimension sequences of unit steps to the right or left span a lattice, while in two or higher dimensions any point can be reached by unit steps.

As in the three-dimensional case, the path integral can be written in a special form provided one chooses  $\eta$  and the representation of the Dirac matrices appropriately. Now one choice for  $\beta$  and  $\alpha$  is simply the restriction of the Weyl representation (21), e.g. for  $d=1$  we have  $\{\beta, \alpha_z\}$  and for  $d=2$ ,  $\{\beta, \alpha_x, \alpha_y\}$ . With this representation and  $\eta = 1/d$ , the spinor chain form (25) for the path integral can be taken over directly, with the allowed spinors restricted to  $\{(1, 0), (0, 1)\}$  for  $d=1$  and  $\{2^{-1/2}(e^{-i\phi/2}, e^{i\phi/2}) : 0 \leq \phi < 2\pi\}$  for  $d=2$ . Different values of  $\eta = \varepsilon/a$  specify different step speeds, so that the lower-dimensional cases are not simply obtained by restricting the spinors to some subset of spin space. The connection between spinors and spacetime vectors must also be modified.

In one space dimension the spinors play no role, since unless the same spinor appears at every step the amplitude will be zero. Thus for a one-dimensional Dirac particle, the only freedom is the chirality at each step, which is equivalent to the direction of motion. Furthermore, the steps are the speed of light ( $\eta = 1$ ), so we have recovered Feynman's checkerboard rule (Feynman and Hibbs 1965). Strictly speaking, we really have two copies of Feynman's rule, one for the spinor  $(1 \ 0)$  and one for  $(0 \ 1)$ . To obtain an irreducible representation in one space dimension one should represent  $\beta$  and  $\alpha$  with  $2 \times 2$  rather than  $4 \times 4$  matrices. For example,  $\beta = \sigma_x$  and  $\alpha = \sigma_z$  will do the job.

In two dimensions the inner product of two spinors is just  $\cos[\frac{1}{2}(\phi' - \phi)]$ , where  $\phi'$  and  $\phi$  are the angles defining the corresponding unit vectors. It is also possible to use a two-component wavefunction in two spatial dimensions by choosing for example the  $2 \times 2$  matrix representation  $\beta = \sigma_z$  and  $\alpha = (\sigma_x, \sigma_y)$ . With this representation of course the spinor chain form (25) is not correct, since the mass term acts within the single two-component spin space.

### 3. Stochastic aspects of the path integral

An initially surprising feature of the path integral (13) is that the minimum allowable step speed is actually greater than the speed of light when the space has more than one dimension. Two questions are asking to be answered; 'Why cannot the steps be at the speed of light?' and 'Given that the steps are faster than light, how does it happen that the maximum speed of propagation is nevertheless the speed of light?'. In §§ 3.1–3.3, we attempt to answer these questions by examining some stochastic aspects of the path integral. Section 3.4 discusses the effect of mass and the passage to the non-relativistic limit.

### 3.1. One dimension

In one spatial dimension the massless propagation matrix (cf (27)) with  $\alpha = \sigma_z$  takes the form

$$\begin{pmatrix} 1 + \eta n & 0 \\ 0 & 1 - \eta n \end{pmatrix}.$$

If  $p$  and  $q$  are the number of steps to the right and left, the amplitude for a net displacement of  $M = p - q$  steps in a total of  $N = p + q$  steps is given by

$$\binom{N}{p} \left(\frac{1 + \eta}{2}\right)^p \left(\frac{1 - \eta}{2}\right)^q. \tag{28}$$

The two signs correspond to 'right- and left-handed' particles. The stability condition is  $\eta^2 \leq 1$  so we may choose  $\eta = 1$ , which shows immediately that a massless right(left)-handed particle can move only at the speed of light to the right(left), since if  $q \neq 0$  ( $p \neq 0$ ) the amplitude is zero.

It is instructive to see how things work out if we choose  $\eta < 1$ , corresponding to steps faster than light. Then (28) is identical to the *probability* distribution for a one-dimensional random walk with drift. In a statistical mechanics application, the drift might arise from an external field, for example, and take the form  $\eta = a/a_0$  where  $a$  is the step length and  $a_0$  is a constant with the dimension of length. The distribution after  $N$  steps would be approximately described by a Gaussian of width  $\sim N^{1/2}a$  drifting at a speed  $\eta a/\epsilon = D/a_0$ , where  $D = a^2/\epsilon$  is the (fixed) diffusion constant. In the case of the Dirac particle on the other hand, the step speed  $a/\epsilon = c/\eta > c$  fixed, the drift speed is  $\eta a/\epsilon = c$ , and the width of the Gaussian goes to zero in the limit  $\epsilon, a \rightarrow 0$  since  $N = (t - t')/\epsilon$ .

A partial answer can now be given to the second question asked at the beginning of this section. although there is a greater amplitude to continue in the same direction on a given path, there are so many more paths with more turns that the peak of the distribution (28) occurs for smaller values of  $M$ . It is a balancing of two exponential factors that picks out the form of the drifting distribution. Then, in the particular limit involved in our path integral, the distribution becomes a  $\delta$  function drifting at the speed of light. This is only a partial answer since it ignores the features that arise in a space of higher dimension.

### 3.2. Two and three dimensions

In a space of two or more dimensions the particle has a continuous infinity of unit vectors to choose from at each step, and all but possibly one of these have a non-zero amplitude of order unity to follow the previous step. This situation is analogous to the one-dimensional case with  $\eta < 1$ , so it is plausible that in the limit of infinitesimal steps there is a vanishing amplitude to continue in a straight line at the step speed. It seems therefore that one does not have the option to choose steps at the speed of light. As a matter of fact, the stability of our finite difference scheme requires that the step speed be at least  $2^{1/2}c$  or  $3^{1/2}c$  in a space of two or three dimensions respectively (cf §§ 2.3, 2.7).

Other than the stability calculation itself one can offer the following quasi-physical explanation for the value of the minimum step speed  $d^{1/2}c$ . In the usual theory of the

Dirac electron, the velocity operator is  $c\alpha$ , so that the expectation value of the squared speed in *any* state  $\psi$  is given by

$$c^2\langle\psi|\alpha \cdot \alpha|\psi\rangle = dc^2. \tag{29}$$

We emphasise that this is a standard quantum mechanical calculation, not related to our path integral formulation. The result (29) has often been either regarded as an embarrassment or disregarded on the basis that  $c\alpha$  is not the time derivative of the ‘relevant’ position operator. In the present context, however, the operator  $x$  whose eigenvalues are the coordinates of spatial points *is* the relevant position operator, and (29) makes it plausible that any stochastic process underlying the Dirac equation must consist of steps of speed at least  $d^{1/2}c$ —for otherwise how could the mean square speed be  $dc^2$ ?

### 3.3. Spinor chain drift mechanism

As in the one-dimensional case with  $\eta < 1$  a drift mechanism is evidently operating and again there is a stochastic approach to the light cone. Let us examine the drift mechanism in the spinor chain form of the path integral, with step speeds  $c$ ,  $2c$  and  $3c$  in 1, 2 and 3 spatial dimensions respectively.

The amplitude for a spinor  $\nu$  to be followed by  $\nu'$  has the form  $\langle\nu'|\nu\rangle = \cos\frac{1}{2}\omega e^{i\zeta}$ , where  $\omega$  is the angle between the corresponding unit vectors and  $\zeta$  is a non-geometrical phase angle (see discussion in § 2.6). The factor  $\cos\frac{1}{2}\omega$  is equal to unity when the corresponding unit vectors are parallel and zero when they are antiparallel ( $\omega = \pi$ ). Hence there is a correlation in direction for successive steps and this produces the drift.

In one dimension only two directions are available so that  $\cos\frac{1}{2}\omega$  is either zero or unity. There is complete correlation and since each step is at the speed of light the ‘drift’ is at that speed.

To ascertain the drift in two or three dimensions one may calculate the displacement expected after one step. According to the basic rule of quantum mechanics  $|\langle\nu'|\nu\rangle|^2 = \cos^2\frac{1}{2}\omega$  gives the *probability* for  $\nu'$  to succeed  $\nu$ . Thus if the initial step in three dimensions is parallel to the unit vector  $\hat{z}$ , the expected displacement during the next step is given by

$$(2\pi)^{-1} \int d\Omega (\cos^2\frac{1}{2}\theta)\hat{n}a = \frac{1}{3}\hat{z}a.$$

(The probability function  $\cos^2\frac{1}{2}\theta$  is normalised on the sphere with the measure  $d\Omega/2\pi$ . This is precisely the measure of one step in the path integral (cf § 2.5).)

Since  $a = 3c\epsilon$ , and the duration of one step is  $\epsilon$ , it follows that the expected displacement or drift is in the  $\hat{z}$  direction at the speed of light. A similar computation shows that also in two dimensions the drift is at the speed of light. This is not the whole story, however. By itself, the weighting toward the forward direction of the previous step is insufficient to produce a long-range correlation. The phase  $e^{i\zeta}$  in the transition amplitude  $\langle\nu'|\nu\rangle = \cos\frac{1}{2}\omega e^{i\zeta}$  plays a crucial role. Furthermore, the behaviour in two and three dimensions is evidently qualitatively different. Indeed, the retarded propagator for the massless wave equation in a space of even dimension is non-zero in the interior of the forward light cone, whereas in a space of odd dimension  $\geq 3$  it is non-zero only on the light cone (see e.g. Courant and Hilbert 1962).

3.4. Mass and the non-relativistic limit

The path integral (25) shows that there is an amplitude  $i\epsilon m$  to switch chirality at each step of a spinor chain. By a chirality switch the electron can reverse its direction of motion while repeating the same spinor. The persistence in spatial direction for successive steps is thus destroyed every time the mass term acts.

From the Dirac equation Hamiltonian form,  $i\partial_t\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi$ , it is seen that for non-relativistic momenta  $p \ll m$ , the time scale for one chirality switch is  $\Delta t \sim 1/m = \hbar/mc^2$ . In path integral language we may say that in between switches the electron propagates as if it were massless, travelling at the speed of light. It is thus plausible that in the non-relativistic limit an electron may be described by a sequence of *uncorrelated* steps of duration  $\Delta t$  and speed  $\Delta x/\Delta t = c$ . This leads to a picture of a 'diffusing' particle with 'diffusion constant'  $(\Delta x)^2/\Delta t \sim \hbar/m$ , which is consistent with the usual non-relativistic path integral or Schrödinger equation. This stochastic account of the passage to the non-relativistic description (in the one-dimensional case) is discussed in Jacobson and Schulman (1984).

4. Propagation in an external gauge potential

Replacing the derivative  $\partial_\mu$  in the Dirac equation by a gauge covariant derivative  $D_\mu$  one produces an equation governing the motion of an electron (or a gauge multiplet) minimally coupled to a gauge potential. By modifying the derivation of § 2.2 we obtain the form of the path integral for such situations. It turns out that the effect of a gauge potential is simply to multiply the amplitude for each path by the transport operator  $\mathcal{P} \exp(-ig \int_C A_\mu dx^\mu)$  for that path, and from this rule we derive a perturbation expansion for the propagator that agrees with the usual formula.

4.1. Electromagnetic potential

In the presence of an electromagnetic potential  $A_\mu$ , the Dirac equation (7) becomes

$$\gamma^\mu D_\mu \gamma = -im \psi$$

with  $D_\mu = \partial_\mu + ieA_\mu$ ;  $e$  is the electronic charge. Following the steps leading from (7) to (9) one obtains

$$i\epsilon m \beta \psi = \sum (1 + 3\eta \boldsymbol{\alpha} \cdot \hat{\mathbf{n}})(-n^\nu D_\nu \psi). \tag{30}$$

Now we make the gauge covariant finite difference approximation

$$-n^\nu D_\nu \psi = \exp\left(-ie \int_{x-n}^x A_\mu dx^\mu\right) \psi(x-n) - \psi(x) + O(\epsilon^2). \tag{31}$$

Substituting (31) in (30) we obtain

$$\psi(x) = (1 - i\epsilon m \beta) \sum (1 + 3\eta \boldsymbol{\alpha} \cdot \hat{\mathbf{n}}) \exp\left(-ie \int_{x-n}^x A_\mu dx^\mu\right) \psi(x-n) + O(\epsilon^2). \tag{32}$$

Equation (32) indicates that according to the Dirac equation, as  $\epsilon \rightarrow 0$  the matrix

$$M^\Lambda(x, \hat{\mathbf{n}}) = M_\epsilon(\hat{\mathbf{n}}) \exp\left(-ie \int_x^{x+n} A_\mu dx^\mu\right) \tag{33}$$



propagates the spinor amplitude from  $x$  to  $x + (\hat{n}_a, \varepsilon)$  in the presence of the electromagnetic potential  $A_\mu$ . Assuming the exponential term in  $A_\mu$  does not affect the stability of our finite difference scheme, it follows from the results of § 2.3 that when  $\eta^2 < \frac{1}{3}$  the propagator  $K^A$  in the presence of the potential is given by  $\lim_{\varepsilon \rightarrow 0} K_\varepsilon^A$  with

$$\begin{aligned}
 K_\varepsilon^A(x, x') &= \int \Pi(d\Omega_i/4\pi) M_\varepsilon^A(x - n_N, \hat{n}_N) \dots M_\varepsilon^A(x', \hat{n}_1) \\
 &= \int \Pi(d\Omega_i/4\pi) M_\varepsilon(\hat{n}_N) \dots M_\varepsilon(\hat{n}_1) \exp\left(-ie \int_C A_\mu dx^\mu\right) \tag{34}
 \end{aligned}$$

where we have combined all the phase factors into one integral of  $A_\mu$  over the polygonal spacetime path  $C$  connecting  $(x', t')$  to  $(x, t)$  via the vertices  $(x', t') + \sum_{i=1}^j (\hat{n}_i, \varepsilon)$  for  $j \leq N = (t - t')/\varepsilon$ . In spinor chain language, the propagator connecting states of definite chirality is given (cf (25)) by

$$[K_\varepsilon^A]_{x \searrow x_0} = \int \Pi(d\Omega_i/2\pi) \sum_{x_1, \dots, x_{N-1}} |\nu_N\rangle \langle \nu_N| \dots |\nu_1\rangle \langle \nu_1| (ie m)^R \exp\left(-ie \int_C A_\mu dx^\mu\right). \tag{35}$$

The effect of an external electromagnetic potential is thus simply to multiply the amplitude for a given path by the phase change along the path. This result stands in sharp contrast to the usual treatment of a Dirac electron propagating in a potential (see, however, Feynman (1950, § 6) for a related observation). Commonly, the propagator  $K^A$  is defined as a solution to the equation  $[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]K^A(x, x') = i\gamma^0\delta^4(x, x')$ . The important difference is that  $A_\mu$  appears only in the combination  $\gamma^\mu A_\mu = \mathcal{A}$ , which is also the case in the usual perturbation expansion for  $K^A$

$$\begin{aligned}
 K^A(x, x') &= K(x, x') + \int d^4x_1 K(x, x_1)[-ie\gamma^0\mathcal{A}(x_1)]K(x_1, x) \\
 &\quad + \iint d^4x_2 d^4x_1 K(x, x_2)[-ie\gamma^0\mathcal{A}(x_2)]K(x_2, x_1)[-ie\gamma^0\mathcal{A}(x_1)]K(x_1, x') \\
 &\quad + \dots \tag{36}
 \end{aligned}$$

We shall now see how the expansion (36) arises from the path integral (34) or (35).

#### 4.2. Perturbation expansion

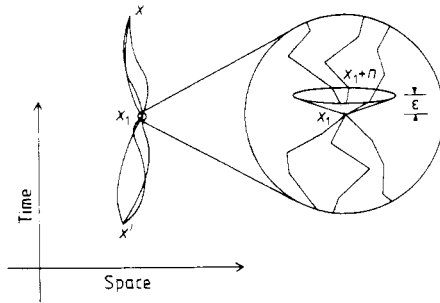
The propagation matrix (33) may be expanded in  $\varepsilon$  as

$$M_\varepsilon^A(x, \hat{n}) = M_\varepsilon(\hat{n})[1 - ieA_\mu(x)n^\mu] + O(\varepsilon^2). \tag{37}$$

In the  $\varepsilon \rightarrow 0$  limit the term  $O(\varepsilon^2)$  will contribute nothing, and we drop it. The expression for  $K_\varepsilon^A$  (cf (34)) is then a sum of products of  $N = (t - t')/\varepsilon$  matrices of the form (37), and we may break up this sum into partial sums  $K_\varepsilon^A = K_\varepsilon^{(0)} + K_\varepsilon^{(1)} + K_\varepsilon^{(2)} + \dots$ , so that the potential appears exactly  $p$  times in each summand of  $K_\varepsilon^{(p)}$ .  $K_\varepsilon^{(0)}$  is just the free propagator,  $K_\varepsilon$ .

Consider all the summands in  $K_\varepsilon^{(1)}$  for which the potential acts at the spacetime point  $x_1$  (see figure 2). We can perform the sum over these terms to obtain

$$\sum_{\hat{n}} K(x, x_1 + (\hat{n}_a, \varepsilon)) M_\varepsilon(\hat{n})[-ieA_\mu(x_1)n^\mu] K(x_1, x')$$



**Figure 2.** Spacetime diagram of three paths contributing to first-order scattering from an external gauge potential  $A_\mu$  at  $x_1$ . Expanded view shows the cone of steps leading away from  $x_1$ . Summing the propagation matrix  $M_\epsilon(\hat{n})\mathcal{P} \exp(-ig \int_{x_1}^{x_1+n} A_\mu dx^\mu)$  over this cone yields  $1 - i\epsilon\gamma^0 \mathcal{A}(x_1)$ , giving rise to the usual perturbation expansion.

since the sum over paths from  $x'$  to  $x_1$  just gives the free propagator, and the same for paths from  $x_1 + (\hat{n}\alpha, \epsilon)$  to  $x$ . There remains a single sum (integral) over the cone of steps leading away from  $x_1$ . Since  $K(x, x_1 + (\hat{n}\alpha, \epsilon)) = K(x, x_1) + O(\epsilon)$ , and  $[-ieA_\mu(x_1)n^\mu]$  is already  $O(\epsilon)$ , we may factor the last propagator out of the sum dropping the higher-order terms, to obtain

$$K(x, x_1) \left( \sum_{\hat{n}} M_\epsilon(\hat{n}) [-ieA_\mu(x_1)n^\mu] \right) K(x_1, x'). \tag{38}$$

Now  $\sum_{\hat{n}} M_\epsilon(\hat{n})n^\mu = (1 - i\epsilon m\beta) \sum_{\hat{n}} (1 + 3\eta\alpha \cdot \hat{n})(\hat{n}\alpha, \epsilon) = \epsilon\gamma^0\gamma^\mu + O(\epsilon^2)$ , so that (38) becomes

$$K(x, x_1) [-ie\epsilon\gamma^0 \mathcal{A}(x_1)] K(x_1, x').$$

Finally we integrate over the positron of the spacetime point  $x_1$  to obtain

$$K_\epsilon^{(1)}(x, x') = \int d^4x_1 K(x, x_1) [-ie\gamma^0 \mathcal{A}(x_1)] K(x_1, x')$$

(the sum over times being treated as an integral with  $\epsilon$  playing the role of  $dt$ ). The higher-order terms  $K_\epsilon^{(p)}$  are similarly obtained, and by this procedure the usual perturbation expansion (36) is recovered.

### 4.3. Non-Abelian gauge potential

Everything we have done can be directly extended to the case where the Dirac wavefunction  $\psi$  is a multiplet of  $n$  four-component spinors transforming under an  $n$ -dimensional unitary representation of a non-Abelian internal symmetry group  $G$ . The Dirac equation becomes

$$\gamma^\mu D_\mu \psi = -im \psi$$

with  $D_\mu = \partial_\mu + igA_\mu$ , where the gauge potential  $A_\mu(x)$  is now a matrix in the Lie algebra of the representation of  $G$ . In the absence of interaction each of the  $n$  spinors in  $\psi$  independently satisfies the ordinary Dirac equation, while an external gauge potential ‘rotates’ the  $n$ -tuple as the particle propagates. A calculation virtually identical to the electromagnetic case shows that the propagator can be written as a path integral in

the form  $\lim_{\epsilon \rightarrow 0} K_\epsilon^A$  with

$$K_\epsilon^A(x, x') = \int \Pi(d\Omega_i/4\pi) M_\epsilon(\hat{n}_N) \dots M_\epsilon(\hat{n}_1) \mathcal{P} \exp\left(-ig \int_C A_\mu dx^\mu\right) \quad (39)$$

where  $M_\epsilon(\hat{n})$  is as in (12) (multiplied by the identity in the  $n$ -dimensional 'internal' space) and  $\mathcal{P} \exp(-ig \int_C A_\mu dx^\mu)$  is, in geometrical terms, the path-ordered parallel transport operator that carries an  $n$ -tuple along the path  $C$  from  $x'$  to  $x$ . The amplitude for a transition from  $\psi_i$  to  $\psi_f$  along  $C$  is thus obtained by first rotating  $\psi_i$  to  $\psi'_i \equiv \mathcal{P} \exp(-ig \int_C A_\nu dx^\nu) \psi_i$  and then computing the amplitude for a transition  $\psi'_i \rightarrow \psi_f$  as in the free case.  $\psi'_i$  transforms under gauge transformations as an  $n$ -tuple at  $x$ , which neatly shows that the transition amplitude is gauge invariant. Finally, the perturbation expansion for  $K^A$  arises in precisely the same manner as in the electromagnetic case.

## 5. Discussion

It has been shown that two-component spinors and a binary chirality variable suffice to describe the propagation of a free Dirac particle. The process of propagation in spacetime is reduced to finite concatenations of spin states, the amplitude rule and the spacetime displacement being expressed purely in terms of the quantum algebra of spinors (two-dimensional Hilbert space).

To the pregeometrically inclined this is a striking and suggestive observation. If the duration  $\epsilon$  of one spinor link is taken to be a small but finite quantum of time, all spinor chains will have a finite number of links. Assuming it could be extended to deal with quantised interactions, the spinor chain language would thus provide an alternative to the problematic concept of point particle and avoid the attendant divergences of ordinary continuum quantum field theory<sup>†</sup>.

With  $\epsilon > 0$  the formalism requires a new *interpretation*, since unitarity on the usual electron Hilbert space  $\mathcal{H}_{\text{space}} \times \mathcal{H}_{\text{spin}}$  is lost. In Jacobson (1983) a provisional interpretation for  $\epsilon > 0$  is established, and the emergence of unitarity on  $\mathcal{H}_{\text{space}} \times \mathcal{H}_{\text{spin}}$  in the limit  $\epsilon \rightarrow 0$  is analysed.

Interaction with an external gauge potential has been treated, but of course it requires explicit use of spacetime coordinates to specify the potential. The challenge is thus to describe quantised interactions in the spinor chain language with  $\epsilon > 0$ , the scale of  $\epsilon$  being set ultimately by experiment. In processes which are purely electromagnetic, QED has been confirmed down to scales  $\sim \hbar/150 \text{ GeV} \sim 10^{-16} \text{ cm}/c$  (see Branson 1981). It is not clear however whether this implies that  $\epsilon$  must necessarily be less than  $10^{-16} \text{ cm}/c$ , since one could imagine that by selectively looking at purely electromagnetic processes QED would continue to give good results below the scale of  $\epsilon$ . Care needs to be taken in interpreting experiments, since taking  $\epsilon > 0$  does not involve giving the electron any additional structure.

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<sup>†</sup> Finkelstein (1969, 1972a, b, 1974; Finkelstein *et al* 1974) has previously suggested a similar model of the electron as part of the 'spacetime code' investigations.

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